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# The torsionless affine field theory of Einstein

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**Abstract.** The affine field theory of a symmetric connection is based on a more general variational principle than was used by Einstein. The action density is a function of the Ricci tensor and a covariant vector field $A_i$, which is determined by the connection. The terms in the field equations which depend explicitly on $A_i$ are interpreted as charge-current 4-vectors. The field equations give a generalization of Mie's theory of electrodynamics, and in a special case they correspond to Born's non-linear equations of electrodynamics. A connection with Maxwell's equations is also established in this case.

The static spherically symmetric field of an electron is described, and an interpretation of the solutions, which ascribes a structure to the electron, is briefly discussed.

## 1. Introduction

In an early attempt to find a geometrical origin for gravitation and electromagnetism Einstein (1923) formulated a field theory based on the law of parallel displacement

$$\mathrm{d}B^i = -\Gamma^i_{jk} B^j \, \mathrm{d}x^k \tag{1}$$

where $x^i$ ($i = 1, 2, 3, 4$) are the coordinates of a four-dimensional affine space. $B^i$ is a contravariant vector, and $\Gamma^i_{jk}$ are coefficients of connection, which are symmetric for the interchange of $j$ and $k$. Although the theory has mathematical elegance, it has not so far led to the discovery of any physically significant results. Eddington (1930) had the opinion that the theory did not lead along the direct route of real physical progress. Schrödinger also investigated the theory (Schrödinger 1943, 1944, 1947), but later abandoned it in favour of the much more intricate theory based on a connection with torsion.

In the usual development of the symmetric theory (Eddington 1930, p. 257, Schrödinger 1943) it is assumed that the action density is a function only of the components $L_{ij}$ of the generalized Ricci tensor. The field equations then include terms involving the 4-potentials and the equations satisfied by the potentials themselves are of the Proca type. In § 2 the theory is developed for a more general action density $\mathfrak{A}$ which is a function of $L_{ij}$ and a 4-potential $A_i$. Although Einstein's equations are included as a special case, more interesting conclusions are reached in the general case by regarding the terms which depend explicitly on $A_i$ as charge-current vectors, which are sources of the fields. In § 3 it is shown that the equations give a generalization of Mie's theory of the electromagnetic field (Mie 1912, 1913, Weyl 1950, Pauli 1958).

The metric tensor used in § 3 to find the connection with Mie's theory is the usual tensor $g_{ij}$, which is defined through partial derivatives of the action density (Schrödinger 1943). In § 4 an alternative description of the field equations is given in a Riemannian space–time with metric tensor $h_{ij}$, which is the Ricci tensor of $g_{ij}$. It is thus found that a special action density $\mathfrak{A}$ leads to a generalization of Maxwell's equations. An equivalent description of the equations for this case, with $g_{ij}$ as the metric tensor, leads to a generalization of Born and Infeld's electrodynamics. The Born–Infeld generalization was first obtained by Schrödinger (1943), and has also been considered by Cornish (1962), who expressed doubt about the validity of Schrödinger's method of deriving the equations, because he did not determine the form of $\mathfrak{A}$. It was shown by Leopold-George (1965) that a simple form of $\mathfrak{A}$ existed which led to Schrödinger's equations, but it is now found that, in addition to obtaining Schrödinger's equations from the variational principle, one obtains another different set of equations. An important matter to investigate is whether physical significance should be attached to solutions of both sets of equations.

In the theory of Born and Infeld (1934) there were two alternative descriptions of the field of a charged particle. In the one it was found that the electric field was finite at the particle and there was a distribution of charge throughout space which was concentrated near the centre of symmetry of the field, and in the other the electric field became infinite at the charge, and the form of the electrodynamic equations was similar to that of Mie's theory. In the generalization found from the symmetric affine theory the charges are situated at singularities of the fields, and the energy of the electromagnetic field is determined from an affine tensor density $\mathfrak{T}^i_j$. When $g_{ij}$ is the metric tensor, $\mathfrak{T}^i_j$ gives the energy tensor density associated with Mie's theory and also determines the stress energy and momentum tensor in Einstein's gravitational field equations, but when $h_{ij}$ is the metric tensor $\mathfrak{T}^i_j$ has to be supplemented by a non-Maxwellian stress tensor in the latter equations.

In § 6 the static spherically symmetric field of an electron is discussed. One interpretation of the solutions leads to the conclusion that the electron has a structure. Although the theory of the electron belongs to the quantum domain, it is still felt by some that interest attaches to a crude classical model of an electron (Rohrlich 1965). By assuming that the history of an electron is a world tube in space–time, which is bounded by a hypersurface on which some components of the fields become singular, it is found that the solutions of the two sets of field equations, which were discussed above, give the fields in the interior and exterior regions of the tube. Also the property of having a finite field energy, which was found for the Born electron (Born 1934), exists in this generalized theory. These matters are only briefly discussed and the problem of stability is not considered.

## 2. The general affine theory

From the law of parallel displacement (1) it follows that there exists a tensor (Eisenhart 1927)

$$L^i_{jkl} = -\frac{\partial \Gamma^i_{jk}}{\partial x^l} + \frac{\partial \Gamma^i_{jl}}{\partial x^k} + \Gamma^i_{mk}\,\Gamma^m_{jl} - \Gamma^i_{ml}\,\Gamma^m_{jk}. \tag{2}$$

Contracting with respect to $i$ and $l$ gives the Ricci tensor

$$L_{jk} = -\frac{\partial \Gamma^l_{jk}}{\partial x^l} + \frac{\partial \Gamma^l_{jl}}{\partial x^k} + \Gamma^l_{mk}\,\Gamma^m_{jl} - \Gamma^l_{ml}\,\Gamma^m_{jk} \tag{3}$$

which has the symmetric and antisymmetric parts

$$L_{\underline{jk}} = -\frac{\partial \Gamma^l_{jk}}{\partial x^l} + \frac{1}{2}\left(\frac{\partial \Gamma^l_{jl}}{\partial x^k} + \frac{\partial \Gamma^l_{kl}}{\partial x^j}\right) + \Gamma^l_{mk}\,\Gamma^m_{jl} - \Gamma^l_{ml}\,\Gamma^m_{jk} \tag{4}$$

$$L_{\check{jk}} = \frac{1}{2}\left(\frac{\partial \Gamma^l_{jl}}{\partial x^k} - \frac{\partial \Gamma^l_{kl}}{\partial x^j}\right) = -f_{jk} \tag{5}$$

the last equation defining the antisymmetric tensor $f_{jk}$.

Let $A_i$ be a covariant vector field in a region of space $\mathscr{R}$ and let $\mathfrak{A}\,(L_{\underline{ij}}, f_{ij}, A_i)$ be a scalar density which is a function of $L_{\underline{ij}}, f_{ij}$ and $A_i$. If $\Omega$ is a scalar function defined over $\mathscr{R}$, which can be expressed in the form $\frac{1}{2}\ln\phi$, where $\phi$ is a scalar density, then it is a consequence of the transformation properties of the $\Gamma^l_{il}$ (Eisenhart 1927, p. 25) that $\frac{1}{2}\Gamma^l_{il} - \Omega_{,i}$ defines a covariant vector field. We wish to set up a variational principle for the field equations, which will also give the equality of the above two covariant vector fields. Let us consider the stationary principle

$$\delta\int\{\mathfrak{A}(L_{\underline{ij}}, f_{ij}, A_i) + \mathfrak{Q}^i(A_i - \tfrac{1}{2}\Gamma^l_{il} + \Omega_{,i})\}\,\mathrm{d}\tau = 0 \tag{6}$$

where $\mathfrak{Q}^i$ is a contravariant vector density and $\mathrm{d}\tau = \mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3\,\mathrm{d}x^4$, for arbitrary variations $\delta\Gamma^i_{jk}$, $\delta A_i$, $\delta\mathfrak{Q}^i$, $\delta\Omega$, which vanish on the boundary of $\mathscr{R}$. Defining tensor and vector densities $\mathfrak{g}^{ij}$, $\mathfrak{p}^{ij}$, $\mathfrak{j}^{i}$ by the equations

$$\mathfrak{g}^{ij} = 2\frac{\partial\mathfrak{A}}{\partial L_{\underline{ij}}} \tag{7}$$

$$\mathfrak{p}^{ij} = -2\frac{\partial\mathfrak{A}}{\partial f_{ij}} \tag{8}$$

$$\mathfrak{J}^{i} = -\frac{\partial\mathfrak{A}}{\partial A_{i}} \tag{9}$$

and neglecting a surface integral over the boundary, we find that (6) reduces to

$$\int(\mathfrak{g}^{ij}\delta L_{\underline{ij}} + \mathfrak{p}^{ij}\delta L_{\check{ij}} - \mathfrak{J}^{i}\delta\Gamma^{l}_{il})\,d\tau = 0 \tag{10}$$

together with the equations $\mathfrak{Q}^{i} = \mathfrak{J}^{i}$ and

$$\mathfrak{J}^{i}_{,i} = 0 \tag{11}$$

$$A_{j} = \tfrac{1}{2}\Gamma^{l}_{jl} - \Omega_{,j}. \tag{12}$$

The comma denotes partial differentiation. Substituting from equations (4) and (5) in (10), carrying out some integrations and omitting surface integrals, which will be zero because the variations are assumed to vanish on the boundary, we find that

$$(\mathfrak{g}^{ij})_{k} + \tfrac{1}{3}(\mathfrak{j}^{i} + \mathfrak{J}^{i})\delta^{j}_{k} + \tfrac{1}{3}(\mathfrak{j}^{j} + \mathfrak{J}^{j})\delta^{i}_{k} = 0 \tag{13}$$

where $(\mathfrak{g}^{ij})_{k}$ is the affine derivative

$$(\mathfrak{g}^{ij})_{k} = \frac{\partial\mathfrak{g}^{ij}}{\partial x^{k}} + \Gamma^{i}_{kl}\,\mathfrak{g}^{lj} + \Gamma^{j}_{kl}\,\mathfrak{g}^{il} - \Gamma^{l}_{kl}\,\mathfrak{g}^{ij} \tag{14}$$

and

$$\mathfrak{p}^{ij}_{.j} = \mathfrak{j}^{i} \tag{15}$$

where it will be noted that use has been made of the fact that the affine divergence of a skew symmetric contravariant tensor density is equal to the ordinary divergence. As a consequence of (15) the vector density $\mathfrak{j}^{i}$ satisfies the equation

$$\mathfrak{j}^{i}_{,i} = 0. \tag{16}$$

When $\mathfrak{A}$ is a given function of $L_{ij}, f_{ij}$ and $A_{i}$, the $\mathfrak{g}^{ij}$, $\mathfrak{p}^{ij}$, $\mathfrak{J}^{i}$ may be determined from equations (7), (8) and (9), and then the equations (13), (14) and (15) can be solved for the $\Gamma^{i}_{jk}$. The procedure is straightforward and, although the solution given here is more general than the usual one (Eddington 1930, p. 258), it need only be given in outline.

We associate with the affine space a line element having fundamental tensor $s_{ij}$, which determines a Riemannian space–time of signature $-2$. The Christoffel symbols will be denoted by $\{^{i}_{jk}\}_{s}$, and the covariant derivative of tensors will be denoted by $\underset{s}{|}$ followed by a suffix, and $s$ will stand for the determinant $|s_{ij}|$. Then

$$\mathfrak{g}^{ij}_{\underset{s}{|}k} = \frac{\partial\mathfrak{g}^{ij}}{\partial x^{k}} + \{^{i}_{kl}\}_{s}\,\mathfrak{g}^{lj} + \{^{j}_{kl}\}_{s}\,\mathfrak{g}^{il} - \{^{l}_{kl}\}_{s}\,\mathfrak{g}^{ij}. \tag{17}$$

Let

$$\Gamma^{i}_{jk} = \{^{i}_{jk}\}_{s} + S^{i}_{jk} \tag{18}$$

so that

$$(\mathfrak{g}^{ij})_{k} - \mathfrak{g}^{ij}_{\underset{s}{|}k} = S^{i}_{kl}\,\mathfrak{g}^{lj} + S^{j}_{kl}\,\mathfrak{g}^{il} - S^{l}_{kl}\,\mathfrak{g}^{ij}. \tag{19}$$

The equations (13) can then be expressed in the form

$$\mathfrak{g}^{ij}_{\ |k}{}_{s} + S^{i}_{kl}\,\mathfrak{g}^{lj} + S^{j}_{kl}\,\mathfrak{g}^{il} - S^{l}_{kl}\,\mathfrak{g}^{ij} + \tfrac{1}{3}(\mathfrak{j}^{i}+\mathfrak{J}^{i})\delta^{j}_{k} + \tfrac{1}{3}(\mathfrak{j}^{j}+\mathfrak{J}^{j})\delta^{i}_{k} = 0. \tag{20}$$

We define the tensors $q^{ij}$, $q_{ij}$ by the equations

$$\mathfrak{g}^{ij} = q^{ij}\sqrt{-s}, \qquad q^{ij}q_{ik} = \delta^{j}_{k} \tag{21}$$

and let

$$\mathfrak{j}^{i} = \mathfrak{g}^{il}j_{l}, \qquad \mathfrak{J}^{i} = \mathfrak{g}^{il}J_{l}. \tag{22}$$

Then the equations (20) can be solved for the $S^{i}_{jk}$ by the usual method, and it is found that

$$S^{i}_{jk} = \tfrac{1}{2}q^{il}(q_{kl\underset{s}{|}j} + q_{jl\underset{s}{|}k} - q_{jk\underset{s}{|}l}) - \tfrac{1}{4}q^{lm}(q_{lm\underset{s}{|}j}\delta^{i}_{k} + q_{lm\underset{s}{|}k}\delta^{i}_{j} - q^{in}q_{jk}q_{lm\underset{s}{|}n})$$

$$+\tfrac{1}{6}(j_{j}+J_{j})\delta^{i}_{k} + \tfrac{1}{6}(j_{k}+J_{k})\delta^{i}_{j} - \tfrac{1}{2}(j_{l}+J_{l})q^{il}q_{jk}. \tag{23}$$

Let $\sigma = \frac{1}{2}\ln(\sqrt{-s}/\sqrt{-q})$ and define the symmetric tensor $g_{ij}$ and its associated tensor $g^{ij}$ by the equations

$$g_{ij} = e^{2\sigma}q_{ij}, \qquad g^{ij}g_{ik} = \delta^{j}_{k}. \tag{24}$$

Then from (21) and (24)

$$\mathfrak{g}^{ij} = g^{ij}\sqrt{-g} \tag{25}$$

and from (18), (22), (23) and (24) it is found that

$$\Gamma^{i}_{jk} = \{^{\,i}_{jk}\}_{g} + \tfrac{1}{6}(j_{j}+J_{j})\delta^{i}_{k} + \tfrac{1}{6}(j_{k}+J_{k})\delta^{i}_{j} - \tfrac{1}{2}(j_{m}+J_{m})g^{im}g_{jk} \tag{26}$$

where $\{^{\,i}_{jk}\}_{g}$ are the Christoffel symbols for the tensor $g_{ij}$. Contracting $i$ and $k$ in (26)

$$\Gamma^{l}_{jl} = \frac{\partial}{\partial x^{j}}\ln\sqrt{-g} + \tfrac{1}{3}(j_{j}+J_{j}) \tag{27}$$

and comparing (12) and (27), we have

$$j_{i}+J_{i} = 6A_{i} + 3\,\frac{\partial}{\partial x^{i}}\,(2\Omega - \ln\sqrt{-g}). \tag{28}$$

Since $\Omega = \frac{1}{2}\ln\phi$, where $\phi$ is a scalar density, we shall choose $\phi = \sqrt{-g}$, so that the last term in (28) vanishes. There is, however, an arbitrariness in the choice of $\Omega$ which has the effect of permitting changes of gauge, since the $A_{i}$ can be identified with the electrodynamic 4-potential. From (26) we then find

$$\Gamma^{i}_{jk} = \{^{\,i}_{jk}\}_{g} + A_{j}\delta^{i}_{k} + A_{k}\delta^{i}_{j} - 3A_{m}g^{im}g_{jk}. \tag{29}$$

Using (11), (16) and (25), we find from (28), since the gradient term vanishes,

$$A^{i}_{\ \underset{g}{|}i} = 0. \tag{30}$$

When the expressions (29) are substituted in equations (4) and (5), it is found that

$$L_{ij} = R_{ij} + 6A_{i}A_{j} \tag{31}$$

$$\overline{f_{ij}} = A_{j,i} - A_{i,j} \tag{32}$$

where $R_{ij}$ is the Ricci tensor formed with respect to the $g_{ij}$. Also the equation

$$f_{ij,k} + f_{ki,j} + f_{jk,i} = 0 \tag{32'}$$

is a consequence of equation (32).

In Einstein's theory it is assumed that the action density is a function of $L_{ij}$, $f_{ij}$ only, and then from (9), (22) and (28), $J_i = 0$, $j_i = 6A_i$. When equations (7) and (8) are used to express $L_{\underline{ij}}$, $\mathfrak{p}^{ij}$ as functions of $g_{ij}$, $f_{ij}$ and $A_i$ the equations (15), (30), (31) and (32) give the field equations of the theory.

The inclusion of the covariant vector $A_i$ in the action density appears to be an added complication, but it is shown below that the field equations (15) and (31) do not depend explicitly on the 4-potential $A_i$, except through the sources of the field $\mathfrak{j}^i$.

## 3. The relation to Mie's theory

We assume that equations (7) can be solved to give the components $L_{\underline{ij}}$ as functions of the components $\mathfrak{g}^{ij}$, $f_{ij}$, $A_i$ and that $\mathfrak{A}$ is expressed in terms of these quantities. Then we define $\mathfrak{L} \equiv \mathfrak{L}(\mathfrak{g}^{ij}, f_{ij}, A_i)$, a function of $\mathfrak{g}^{ij}$, $f_{ij}$, $A_i$ by

$$\mathfrak{L} = \tfrac{1}{2}\mathfrak{g}^{ij}R_{ij} - \mathfrak{A}. \tag{33}$$

Using (7), (8), (9), (31) and (33)

$$d\mathfrak{L} = \tfrac{1}{2}R_{ij}\, d\mathfrak{g}^{ij} + \tfrac{1}{2}\mathfrak{p}^{ij}\, df_{ij} + (\mathfrak{J}^i - 6\mathfrak{g}^{ij}A_j)\, dA_i. \tag{34}$$

Hence, from (28) and (34),

$$R_{ij} = 2\frac{\partial \mathfrak{L}}{\partial \mathfrak{g}^{ij}} \tag{35}$$

$$\mathfrak{p}^{ij} = 2\frac{\partial \mathfrak{L}}{\partial f_{ij}} \tag{36}$$

$$\mathfrak{j}^i = -\frac{\partial \mathfrak{L}}{\partial A_i}. \tag{37}$$

We define the tensors

$$F^{ij} = g^{il}g^{jm}f_{lm}, \qquad F^{*ij} = \tfrac{1}{2}\, e^{ijkl}f_{kl}\frac{1}{\sqrt{-g}} \tag{38}$$

where $e^{ijkl}$ is the fourth-order permutation tensor of weight $+1$, and the invariants

$$F = \tfrac{1}{2}F^{ij}f_{ij}, \qquad G = \tfrac{1}{4}F^{*ij}f_{ij} \tag{39}$$

and

$$u = \tfrac{1}{2}g^{ij}A_iA_j. \tag{40}$$

Then, since $\mathfrak{L}$ is a scalar density which is a function of $g_{ij}$, $f_{ij}$, $A_i$, we assume that it can be expressed in the form

$$\mathfrak{L} = \phi(F, G, u)\sqrt{-g} \tag{41}$$

where $\phi$ is an arbitrary function of $F$, $G$ and $u$. Since $u$ is not the only invariant which can be formed from the $A_i$ and the other field tensors, equation (41) does not give the most general form for $\mathfrak{L}$. We find from (37), (40) and (41) that

$$\mathfrak{j}^i = -\mathfrak{g}^{ij}\frac{\partial \phi}{\partial u}A_j. \tag{42}$$

Using (38), (39), (41) and (42), the following identities can be proved:

$$\mathfrak{g}^{lm}\frac{\partial \mathfrak{L}}{\partial \mathfrak{g}^{lm}} + f_{lm}\frac{\partial \mathfrak{L}}{\partial f_{lm}} = \tfrac{1}{2}\mathfrak{j}^lA_l + 2\mathfrak{L} \tag{43}$$

$$-2\mathfrak{g}^{il}\frac{\partial \mathfrak{L}}{\partial \mathfrak{g}^{jl}} + \tfrac{1}{2}\mathfrak{g}^{lm}\frac{\partial \mathfrak{L}}{\partial \mathfrak{g}^{lm}}\delta^i_j = -2f_{jl}\frac{\partial \mathfrak{L}}{\partial f_{il}} + \tfrac{1}{2}f_{lm}\frac{\partial \mathfrak{L}}{\partial f_{lm}}\delta^i_j + \mathfrak{j}^iA_j - \tfrac{1}{4}\mathfrak{j}^lA_l\delta^i_j. \tag{44}$$

From equations (35)

$$(-R^i_j+\tfrac{1}{2}R\delta^i_j)\sqrt{-g} = \mathfrak{T}^i_j \tag{45}$$

where

$$\mathfrak{T}^i_j = -2\mathfrak{g}^{il}\frac{\partial \mathfrak{L}}{\partial \mathfrak{g}^{jl}}+\mathfrak{g}^{lm}\frac{\partial \mathfrak{L}}{\partial \mathfrak{g}^{lm}}\delta^i_j \tag{46}$$

and, using equation (36) and the identities (43) and (44), $\mathfrak{T}^i_j$ can be expressed in the form

$$\mathfrak{T}^i_j = -\mathfrak{p}^{il}f_{jl}+\mathfrak{j}^i A_j+\mathfrak{L}\delta^i_j. \tag{47}$$

On account of equation (45) the conservation equation

$$\mathfrak{T}^l_{j\underset{g}{||}l} = 0 \tag{48}$$

is satisfied.

The equations (15), (16), (32), (32′), (36), (37), (45), (47) and (48) are the equations of Mie, generalized to include the gravitational field (subject to the restriction stated after equation (41)). This interpretation assumes that the fundamental tensor $s_{ij}$ is identified with $g_{ij}$, but it should be noted that the derivation of the equations given above is not dependent on making this assumption.

## 4. The relation to Maxwell's theory

Whereas the equations of Mie were derived from the function $\mathfrak{L}$, we shall now show that the equations of Maxwell come directly from the action density $\mathfrak{A}$. A simple connection between the two theories can, however, only be expected to be found for vacuum fields, on account of the different interpretations which are given to the charge current density $\mathfrak{j}^i$.

Let us define

$$h_{ij} = L_{\underline{ij}}-6A_iA_j \tag{49}$$

so that, from (31), $h_{ij}$ will be the Ricci tensor formed with respect to $g_{ij}$, when the stationary principle (6) is satisfied. Also, by expressing $L_{\underline{ij}}$ as a function of $h_{ij}$, $A_i$ in the action density, we obtain a new function $\mathfrak{h} \equiv \mathfrak{h}(h_{ij}, f_{ij}, A_i)$, where

$$\mathfrak{A} \equiv \mathfrak{h}(h_{ij}, f_{ij}, A_i). \tag{50}$$

From (7), (8), (9), (49) and (50) we find

$$\mathrm{d}\mathfrak{h} = \tfrac{1}{2}\mathfrak{g}^{ij}\,\mathrm{d}h_{ij}-\tfrac{1}{2}\mathfrak{p}^{ij}\,\mathrm{d}f_{ij}+(6\mathfrak{g}^{ij}A_j-\mathfrak{J}^i)\,\mathrm{d}A_i \tag{51}$$

and hence, from (28) and (50), when $\Omega = \tfrac{1}{2}\ln(\sqrt{-g})$

$$\mathfrak{g}^{ij} = 2\frac{\partial \mathfrak{h}}{\partial h_{ij}} \tag{52}$$

$$\mathfrak{p}^{ij} = -2\frac{\partial \mathfrak{h}}{\partial f_{ij}} \tag{53}$$

$$\mathfrak{j}^i = \frac{\partial \mathfrak{h}}{\partial A_i}. \tag{54}$$

The tensor density $\mathfrak{T}^i_j$, defined by (46), can therefore be expressed in the form

$$\mathfrak{T}^i_j = -2h_{jl}\frac{\partial \mathfrak{h}}{\partial h_{il}}+h_{lm}\frac{\partial \mathfrak{h}}{\partial h_{lm}}\delta^i_j \tag{55}$$

by use of (35), (49), (50) and (52), provided the equations (31) are satisfied.

Let us define the tensors

$$f^{ij} = h^{il}h^{jm}f_{lm}, \qquad f^{*ij} = \tfrac{1}{2}\,e^{ijkl}f_{kl}\frac{1}{\sqrt{-h}} \tag{56}$$

and the invariants

$$P = \tfrac{1}{2}f^{ij}f_{ij}, \qquad Q = \tfrac{1}{4}f^{*ij}f_{ij} \tag{57}$$

$$v = \tfrac{1}{2}h^{ij}A_iA_j \tag{58}$$

and assume that $\mathfrak{h}$ can be expressed in the form

$$\mathfrak{h} = \psi(P, Q, v)\sqrt{-h}. \tag{59}$$

Since $v$ is not the only invariant which can be formed from the $A_i$ and the other field tensors, (59) is not the most general form for $\mathfrak{h}$.

We find from (54), (58) and (59)

$$\mathfrak{j}^i = \frac{\partial\psi}{\partial v}h^{ij}A_j\sqrt{-h} \tag{60}$$

and, since $\mathfrak{j}^i$ is then independent of the antisymmetric field $f_{ij}$, except through the invariants $P$ and $Q$, it must be different in this case from the value given by (42). Also we can prove the following identities by use of (56), (57), (58) and (59):

$$h_{lm}\frac{\partial\mathfrak{h}}{\partial h_{lm}}+f_{lm}\frac{\partial\mathfrak{h}}{\partial f_{lm}} \equiv 2\mathfrak{h}-\tfrac{1}{2}\mathfrak{j}^lA_l \tag{61}$$

$$-2h_{jl}\frac{\partial\mathfrak{h}}{\partial h_{il}}+\tfrac{1}{2}h_{lm}\frac{\partial\mathfrak{h}}{\partial h_{lm}}\delta^i_j \equiv 2f_{jl}\frac{\partial\mathfrak{h}}{\partial f_{il}}-\tfrac{1}{2}f_{lm}\frac{\partial\mathfrak{h}}{\partial f_{lm}}\delta^i_j$$

$$+\mathfrak{j}^iA_j-\tfrac{1}{4}\mathfrak{j}^lA_l\delta^i_j. \tag{62}$$

By using these identities to transform the expression for $\mathfrak{T}^i_j$ in (55), we find

$$\mathfrak{T}^i_j = \mathfrak{E}^i_j+\mathfrak{t}^i_j \tag{63}$$

where

$$\mathfrak{E}^i_j = 2f_{jl}\frac{\partial\mathfrak{h}}{\partial f_{il}}-\tfrac{1}{2}f_{lm}\frac{\partial\mathfrak{h}}{\partial f_{lm}}\delta^i_j \tag{64}$$

$$\mathfrak{t}^i_j = \mathfrak{j}^iA_j+\left(\mathfrak{h}-\tfrac{1}{2}f_{lm}\frac{\partial\mathfrak{h}}{\partial f_{lm}}-\tfrac{1}{2}\mathfrak{j}^lA_l\right)\delta^i_j. \tag{65}$$

We shall assume that $h_{ij}$ is the metric tensor. The Einstein tensor in equation (45) is not, in general, equal to the Einstein tensor formed with respect to $h_{ij}$. It differs by terms involving first and second derivatives of $f_{ij}$, and consequently $\mathfrak{T}^i_j$ is not the whole of the stress–energy–momentum density in Einstein's field equations.

*In a vacuum region, where* $\mathfrak{j}^i = 0$, there exists a class of functions $\mathfrak{h}$ for which

$$\mathfrak{T}^l_{j\,\underset{h}{|}\,l} = 0. \tag{66}$$

Expressing the Christoffel symbols $\{^{\,i}_{jk}\}_g$ in terms of the symbols $\{^{\,i}_{jk}\}_h$, by use of equations (18), (23) and (26), we find that

$$\mathfrak{T}^l_{j\,\underset{h}{|}\,l} = \tfrac{1}{2}q^{kl}q_{km\,\underset{h}{|}\,j}(\mathfrak{T}^m_l-\tfrac{1}{2}\mathfrak{T}^n_n\delta^m_l)$$

$$= \tfrac{1}{2}(q^{lm}h_{lm})_{,j}\sqrt{-h}. \tag{67}$$

The last step makes use of equation (45) and depends on the definition of $\mathfrak{T}^i_j$ in terms of partial derivatives of $\mathfrak{L}$. Since charge-current vectors $\mathfrak{j}^i$ represented by (60) cannot, in general, also be represented by (42), we make the restriction $\mathfrak{j}^i = 0$, so that $\mathfrak{L}$ and $\mathfrak{h}$ are independent of $u$ and $v$, respectively. The condition (66) is then satisfied provided

$$\mathfrak{g}^{lm}h_{lm} = 4C\sqrt{-h} \tag{68}$$

where $C$ is a constant. From equations (52), (61) and (68)

$$\mathfrak{h} = C\sqrt{-h}+\tfrac{1}{2}f_{lm}\frac{\partial\mathfrak{h}}{\partial f_{lm}} \tag{69}$$

which has the general solution

$$\mathfrak{h} = \left\{C+P\chi\left(\frac{P}{Q}\right)\right\}\sqrt{-h} \tag{70}$$

where $\chi(P/Q)$ is an arbitrary function of its argument. We consider the case when $\mathfrak{h}$ is a function of $P$ only and take

$$\mathfrak{h} = (1-\tfrac{1}{2}P)\sqrt{-h}. \tag{71}$$

From (53) and (71)

$$\mathfrak{p}^{ij} = f^{ij}\sqrt{-h} \tag{72}$$

and from (15) and (72)

$$\frac{\partial}{dx^j}(f^{ij}\sqrt{-h}) = 0. \tag{73}$$

Also from (63), (64), (65) and (71)

$$\mathfrak{T}^i_j = (-f^{il}f_{jl}+\tfrac{1}{4}f^{lm}f_{lm}\delta^i_j+\delta^i_j)\sqrt{-h} \tag{74}$$

and

$$\mathfrak{E}^i_j = (-f^{il}f_{jl}+\tfrac{1}{4}f^{lm}f_{lm}\delta^i_j)\sqrt{-h}. \tag{75}$$

The equations (32), (66), (73), (74) and (75) are the generalized form of Maxwell's vacuum equations. They need to be supplemented by the gravitational equations, but these are not given here.

From equations (52) and (71), if we write $\mathfrak{E}^i_j = E^i_j\sqrt{-h}$,

$$\mathfrak{g}^{ij} = h^{il}(\delta^j_l-E^j_l)\sqrt{-h} \tag{76}$$

and from equations (30) and (76)

$$\bar{A}^j{}_{\underset{h}{|i}}(\delta^i_j-E^i_j) = 0 \tag{77}$$

where $\bar{A}^i = h^{ij}A_j$. This condition replaces the usual Lorentz condition.

## 5. The generalization of the Born-Infeld theory

We consider the variational principle of § 2 and express the field equations in terms of the field variables $g_{ij}, f_{ij}$. Thus the sources of the fields are assumed to be at singular events outside $\mathscr{R}$, and, from equation (54), $\mathfrak{h}$ is not explicitly dependent on $A_i$. By equating the square roots of the determinants of each side of (76), we find

$$\sqrt{-g} = N^2\sqrt{-h} \tag{78}$$

where

$$N^2 = \epsilon(1-\tfrac{1}{4}P^2-Q^2) \tag{79}$$

and

$$\begin{aligned}\epsilon &= +1 \text{ if } P^2+4Q^2 < 4\\ \epsilon &= -1 \text{ if } P^2+4Q^2 > 4.\end{aligned} \tag{80}$$

From (76) and (78)

$$g^{ij} = N^{-2}h^{il}(\delta^j_l-E^j_l) \tag{81}$$

and in consequence of the identity

$$E^i_l E^l_j \equiv (\tfrac{1}{4}P^2 + Q^2)\delta^i_j \tag{82}$$

we find

$$g_{ij} = \epsilon h_{il}(\delta^l_j + E^l_j). \tag{83}$$

Also from (39), (56) and (81)

$$F = [P\{(1+\tfrac{1}{2}P)^2 + Q^2\} + 4Q^2]N^{-4} \tag{84}$$

$$G = QN^{-2}. \tag{85}$$

Defining

$$w = (1 + 2F - 4G^2)^{1/2} \tag{86}$$

we find that

$$w = \{(1+\tfrac{1}{2}P)^2 + Q^2\}N^{-2} \tag{87}$$

and hence, from (78), (79) and (87),

$$(w+\epsilon)\sqrt{-g} = 2(1+\tfrac{1}{2}P)\sqrt{-h}. \tag{88}$$

We can now derive $\mathfrak{L}$, from (33), (71), (76) and (88), in the form

$$\mathfrak{L} = \tfrac{1}{2}(w+\epsilon)\sqrt{-g} = \tfrac{1}{2}\{(1+2F-4G^2)^{1/2} + \epsilon\}\sqrt{-g} \tag{89}$$

and we derive the field equations from (89) by use of (15), (31), (35), (36) and (49), in the forms

$$R_{ij} = w^{-1}\{f_{il}F^l_j + \tfrac{1}{2}(1+\epsilon w)g_{ij}\} \tag{90}$$

$$\frac{\partial}{\partial x^j}(p^{ij}\sqrt{-g}) = 0 \tag{91}$$

$$p^{ij} = (F^{ij} - 2GF^{*ij})\frac{1}{w} \tag{92}$$

where $\mathfrak{p}^{ij} = p^{ij}\sqrt{-g}$.

The equations (32), (90), (91), (92) are twenty-six equations for the twenty-six field variables $g_{ij}$, $f_{ij}$, $p^{ij}$, $A_i$. In addition, $A_i$ satisfies the Lorentz condition (30). We convert them to gramme–centimetre units (g.c.u.), with the velocity of light unity, in a local frame such that $g_{ij}$ has values $(-1, -1, -1, +1)$. Let $R_0$ be a standard length and $b$ a standard of electric field strength in g.c.u., and transform the coordinates and the field variables thus:

$$x^i \rightarrow \frac{x^i}{R_0}, \qquad f_{ij} \rightarrow \frac{f_{ij}}{\sqrt{2}b}, \qquad A_i \rightarrow \frac{A_i}{\sqrt{2}R_0 b}. \tag{93}$$

It will be assumed that $x^i$, $f_{ij}$, $A_i$ and derived quantities, such as $F$ and $G$, are all measured in the new units, but that $g_{ij}$, $h_{ij}$ have the same values as in natural units. The constant $b$ is identified with Born's constant $b = e/a^2$, where $e$ is the magnitude of the charge of an electron in g.c.u. and $a$ is a length of the order of magnitude of the 'electron radius'. (In c.g.s. units we should have Born's radius $a = e^2/mc^2$, where $c$ is the velocity of light).

The equations (90) can now be written

$$(-R^i_j + \tfrac{1}{2}R\delta^i_j)\sqrt{-g} = 8\pi\gamma\mathfrak{P}^i_j \tag{94}$$

where

$$\mathfrak{P}^i_j = \frac{R_0{}^2 b^2}{2\pi}\mathfrak{T}^i_j = \frac{1}{4\pi w}[-F^{il}f_{jl} + \{F + b^2(1+\epsilon w)\delta^i_j\}]\sqrt{-g} \tag{95}$$

$$\gamma^{-1} = 4b^2 R_0{}^2 \tag{96}$$

and

$$w = \left(1 + \frac{F}{b^2} - \frac{G^2}{b^4}\right)^{1/2}. \tag{97}$$

$\gamma$ is the gravitational constant in g.c.u. and is equal to $c^{-2}\hat{\gamma}$, where $\hat{\gamma}$ is the value in c.g.s. units. The equation (91) is unchanged and (92) can be written

$$p^{ij} = \left(F^{ij} - \frac{GF^{*ij}}{b^2}\right)\frac{1}{w}. \tag{98}$$

In a local coordinate system the components of the antisymmetric tensors can be expressed in terms of the usual space vectors $(\boldsymbol{B}, \boldsymbol{E})$, $(\boldsymbol{H}, \boldsymbol{D})$ according to the following scheme:

$$\begin{array}{ll} (f_{23}, f_{31}, f_{12}) \to \boldsymbol{B}, & (f_{14}, f_{24}, f_{34}) \to \boldsymbol{E} \\ (p^{23}, p^{31}, p^{12}) \to \boldsymbol{H}, & (p^{41}, p^{42}, p^{43}) \to \boldsymbol{D}. \end{array} \tag{99}$$

In the case when $\epsilon = -1$ the equations (32), (91) and (94)–(99) give the generalization of the Born–Infeld theory obtained previously (Gilbert 1964) from a theory based on a semi-symmetric connection. The equations differ from those obtained by Schrödinger (1943) only by the omission of terms depending explicitly on the $A_i$ in equations (91) and (94). We should have obtained Schrödinger's equations if we had defined $h_{ij}$ equal to $\underline{L}_{ij}$ instead of by equation (49), which makes it equal to the Ricci tensor $R_{ij}$.

## 6. The static spherically symmetric field of an electron

A solution of equations (32), (91) and (94)–(98), representing the field of an electron, was found by Schrödinger (1944), in the case $\epsilon = -1$, and there is no essential difference in the method of solution for $\epsilon = +1$. Both cases are considered below.

The metric of space–time, defined by the tensor $g_{ij}$, is assumed to have the form

$$ds^2 = e^{\nu}\, dt^2 - e^{\lambda}\, dr^2 - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\varphi^2 \tag{100}$$

and $p^{ij}$ is assumed to have only the two non-vanishing components $p^{41}$ and $p^{14}$. The solution can then be written

$$f_{41} = -f_{14} = \frac{e}{(a^4+r^4)^{1/2}} \tag{101}$$

$$p^{41} = -p^{14} = -\frac{e}{r^2} \tag{102}$$

$$e^{\nu} = e^{-\lambda} = 1 - \frac{2\gamma\{m(r)+M\}}{r} - \frac{(1+\epsilon)r^2}{6R_0{}^2} \tag{103}$$

where $M$ is an arbitrary constant and

$$m(r) = \tfrac{1}{3}b^2\{r(a^4+r^4)^{1/2} - r^3\} + \tfrac{2}{3}a^4b^2\int_0^r \frac{dr}{(a^4+r^4)^{1/2}}. \tag{104}$$

When $r$ becomes infinite, $m(r)$ has the value

$$m = \frac{2}{3}\frac{e^2}{a}\int_0^{\infty}\frac{dx}{(1+x^4)^{1/2}} \simeq 1{\cdot}24\,\frac{e^2}{a}. \tag{105}$$

When $\epsilon = -1$, $M = 0$ the value of $e^{\nu}$ always differs from unity by less than $10^{-42}$, and the maximum departure occurs at the origin, where there is a singularity (Schrödinger 1944). The gravitational mass $m$ can be shown to be equal to the field energy given by

$$\int \mathfrak{P}_4^4\, d\tau^{(3)} \tag{106}$$

where $d\tau^{(3)} = dr\, d\theta\, d\phi$ and the integral is taken throughout the region $0 \leqslant r \leqslant \infty$, $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \phi \leqslant 2\pi$. A contribution $\frac{3}{4}m$ comes from the term $\mathfrak{E}_4^4$ and $\frac{1}{4}m$ from the term $\mathfrak{t}_4^4$ in equation (63). It has been shown by Leopold-George (1965) that such an electron will respond to the influence of external fields in accordance with classical theory.

It can be shown that the conditions (80) place no restrictions on the values of $r$ for which the solutions with $\epsilon = -1$ and $\epsilon = +1$ are valid. Thus, from the same action density $\mathfrak{h}$, the two different solutions given by equations (101)–(103) are obtained in the same region of space–time. A possible interpretation is that these solutions give the field of an electron under different external conditions. When $\epsilon = +1$, the $g_{ij}$ field is singular when $r \simeq \sqrt{3}R_0$ as well as at $r = 0$, and accordingly one should assume that there will be a spherically symmetric distribution of charge of total amount $e$ at distance $r \simeq \sqrt{3}R_0$ from the electron of charge $-e$. From equation (96) it is found that $R_0 \simeq 10^8$ cm.

A more interesting interpretation is that the fields occur in the interior and exterior of a world tube, which gives the history of an electron having structure, but that they have been described above in an unsuitable coordinate system. We choose a new coordinate $\rho$ in the radial direction, defined by

$$\rho^2 = \tfrac{1}{2}\{(a^4+r^4)^{1/2}+\epsilon r^2\} \tag{107}$$

so that the regions for which $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \phi \leqslant 2\pi$, $r > 0$ and $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \phi \leqslant 2\pi$, $0 < r < r_0$, where $r_0 \simeq \sqrt{3}R_0$, are mapped onto regions having the same ranges of $\theta$, $\phi$, but with $0 < \rho < a/\sqrt{2}$ and $a/\sqrt{2} < \rho < r_0$, respectively. We consider the fields in terms of the coordinates $(\rho, \theta, \phi, t)$ for the theories of Mie and Maxwell, and, for simplicity, we neglect the effect of the gravitational field by assuming $R_0 \to \infty$, so that $\gamma = 0$ and $r_0 \to \infty$. (In a universe with non-vanishing $\gamma$ one must assume that the influence of external fields will prevent the singularity from developing at $\rho = r_0$.) Then we find, from (101), (102) and (103), for Mie's theory

$$\begin{gathered} g_{11} = -\frac{\epsilon(\rho^4+\frac{1}{4}a^4)^2}{\rho^4(\rho^4-\frac{1}{4}a^4)}, \qquad g_{22} = \frac{g_{33}}{\sin^2\theta} = -\epsilon\left(\rho^2-\frac{a^4}{4\rho^2}\right), \qquad g_{44} = +1 \\ f_{41} = -f_{14} = \frac{\epsilon e}{\{\epsilon(\rho^4-\frac{1}{4}a^4)\}^{1/2}} \\ p^{41} = -p^{14} = \frac{-\epsilon e \rho^4}{\{\epsilon(\rho^4-\frac{1}{4}a^4)\}^{1/2}(\rho^4+\frac{1}{4}a^4)} \end{gathered} \tag{108}$$

and for Maxwell's theory

$$\begin{gathered} h_{11} = -\frac{\rho^4+\frac{1}{4}a^4}{\rho^4-\frac{1}{4}a^4}, \qquad h_{22} = \frac{h_{33}}{\sin^2\theta} = -\rho^2, \qquad h_{44} = \frac{\epsilon\rho^4}{\rho^4+\frac{1}{4}a^4} \\ f^{41} = -f^{14} = -\frac{\epsilon e\{\epsilon(\rho^4-\frac{1}{4}a^4)\}^{1/2}}{\rho^4}. \end{gathered} \tag{109}$$

Let us consider the interpretation of these solutions in terms of the sources of the fields which are situated outside $\mathscr{R}$. From either the $p^{ij}$, or the $f^{ij}$ fields and the corresponding divergence equations (15) and (73), we can conclude by use of Gauss's theorem that, at any time, there is a charge $+e$ at the origin of the spatial coordinates, and a charge $-2e$ distributed over the spherical surface $\rho = a/\sqrt{2}$. Since $\mathfrak{P}^i_j$ has been defined independently of the metric, the energy calculated from (106) is the same in both theories, and the energy of the interior field equals that of the exterior field and has the value $m$.

The tensor $g_{ij}$ determines only a flat space–time in $\mathscr{R}$, but the singular nature of the field at $\rho = a/\sqrt{2}$ can be demonstrated by calculating the scalar curvature $R$ before the limit $R_0 \to \infty$ is taken. It is found for the spherically symmetric field given by (101) and (103), by use of equations (90), (93) and (107), that the value in g.c.u. is

$$R = \frac{4}{\epsilon R_0{}^2(1-a^8/16\rho^8)}. \tag{110}$$

Therefore $R$ is singular at $\rho = a/\sqrt{2}$.

For $\rho < a/\sqrt{2}$ the metric of $g_{ij}$ can be transformed into the metric of special relativity, but with the metric of $h_{ij}$ there are changes of the space-like and time-like characters of intervals corresponding to displacements in the radial and the time directions. When one also takes into account the fact that the energy tensor in Einstein's field equations is the Mie tensor (47), the theory of gravitation and electromagnetism is found to have its simplest form in the Einstein–Mie theory of § 3, but it has not been shown that, in general, a suitable action density can be found.

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